1 Upper bound for thin rectangles

Theorem 1. The tile complexity of a self-assembled, just-barely 3D, k × N rectangle at temperature 1 is O

(

N

k 1

) + k

in the abstract tile assembly model.

The following is a proof by construction. Let d =

⌊

k−4 3

⌋ , |bin(m − 1)|, and c =?N umericalV alueOf T heStartCounter?. r = The remainder(k − 4,3), m =

assembly constitutes a d-digit, ( 10 N

)

base-m

1 d, l =

counter. The value of c is chosen such that the counter stops just before reaching a height of N tiles, at which point, the construction is given a flat “roof” and adds a small number of rows to finish reaching a length of N. We define a Gadget Unit as a collection of gadgets with a singular purpose. Gadgets belonging to the same gadget unit will have their figures grouped together, although, since Vertical Column Tiles are present in a majority of these gadget units, they will only be shown in Figure 1.

(a) Up Column Tile

(b) Down Column Tile

Figure 1: Vertical Column Tiles are used throughout the construction to adjust the height of gadget units.

1.1 Seed gadgets

We begin by hard-coding the base of the construction with the Seed unit. It has d columns, where each column represents a digit (most significant digit first) of c in base m, and a collection of geometric bit-bumps on the columns’ east sides encodes the digits into binary. A small “lip” is added on the east side of the Seed gadget in cases where k − 1 is not divisible by 3; this catches the vertical fill tiles at the end of the construction. See Figure 2.

We define the Seed gadgets as followed:

• Create Seed Start(〈seed,col, d,1〉) from the general gadget in Figure 2a.

• For each i = 1,...,d:

– For each j = 1,...,3l − 3:

∗ Create Up Column(〈seed,col, i, j〉,〈seed,col, i, j + 1〉) from the general gadget in Figure 1a.

– Create Seed Msb(〈seed,col, i,3l − 2〉,〈seed,bit, i, l − 1〉) from the general gadget in Figure 2b

if bit(c, m, i, l) = 0 or Figure 2c if bit(c, m, i, l) = 1. – For each j = 1,...,l − 1:

∗ Create Seed Bit(〈seed,bit, i, j〉,〈seed,bit, i, j − 1〉) from the general gadget in Figure 2e

if bit(c, m, i, j) = 0 or Figure 2f if bit(c, m, i, j) = 1.

• For each i = 1,...,d − 1:

– Create Seed Blocker(〈seed,bit,i + 1,0〉,〈seed,col, i,1〉) from the general gadget in Figure 2d.

• Create Seed Lip(〈seed,bit,1,0〉,〈rw,up,1〉) from the general gadget in Figure 2g if r = 0, Figure 2h if r = 1, or Figure 2i if r = 2.

1

(i) Seed Lip 2

Figure 2: The Seed gadget unit.

1.2 Right wall gadgets

After the Seed unit, a Right Wall unit attaches to the vacant north-facing glue without a Right Wall Foundation gadget (all other Right Wall units will have this gadget). The Right Wall unit’s purpose is to initiate a row of Counter units, and then to block the Return gadget so that a new Right Wall unit and subsequent counter row can form westward. Its general gadgets are shown in Figure 3.

We define the Right Wall gadgets as followed:

• Create Right Wall Foundation(〈rw,found〉,〈rw,up,1〉) from the general gadget in Figure 3a.

• For each i = 1,...,3l:

– Create Up Column(〈rw,up,i〉,〈rw,up,i + 1〉) from the general gadget in Figure 1a.

• Create Right Wall Cap(〈rw,up,3l + 1〉,〈rw,down,1〉) from the general gadget in Figure 3b.

• For each i = 1,...,3l − 1:

– Create Down Column(〈rw,down,i〉,〈rw,down,i + 1〉) from the general gadget in Figure 1b.

• Create Right Wall End(〈rw,down,3l〉,〈inc,start〉) from the general gadget in Figure 3c.

(a) Right Wall Foundation

(a) Seed Start

(b) Seed Msb 0

(c) Seed Msb 1

(b) Right Wall Cap

(c) Right Wall End

Figure 3: The Right Wall gadget unit.

2

(d) Seed Blocker

(e) Seed Bit 0

(f) Seed Bit 1

(h) Seed Lip 1

(g) Seed Lip 0

1.3 Counter gadgets

The set of Counter gadget units consist of m units that increment an individual digit of the counter and m units that copy an individual digit of the counter. Each row of the counter has d Counter units and begins by adjoining the west facing glue of the Right Wall gadget with an incrementing unit. Each Counter unit reads over a series of bit-bumps protruding into their row from the preceding Seed unit or counter row. After reading the bit patterns via Guess gadgets, the set of possible Counter units is narrowed to one, and then a bit pattern is either copied or incremented onto the row above by the unit. To elaborate on this “narrowing”, each set of copy and increment units uses the same starting Guess gadgets. Then, depending on which glue of the Counter gadget is blocked or unblocked by the preceding counter row, a gadget will be initiated but from a subset of eligible Counter units with those gadgets in common. The Counter unit that increments m − 1 to 0 is special because its west-facing glue initiates another increment unit and its south-facing glue initiates the End Left Wall unit (whichever glue is not blocked). Other increment units initiate a copy unit with their west-facing glues and a regular Left Wall unit with their south-facing glues. The copy units similarly initiate another copy unit or a Left Wall unit. The gadgets belonging to the Counter units are shown in Figure 4.

We define the Counter gadgets as followed:

• Create Counter Start(〈inc,start〉 ,〈inc,read,0〉,〈inc,read,1〉) from the general gadget in Fig- ure 4a.

• Create Counter Start(〈copy,start〉,〈copy,read,0〉,〈copy,read,1〉) from the general gadget in Fig- ure 4a.

• For each i = 0,...,l − 2 and each u ∈ {0,1}i:

– Create Counter Read(〈inc,read,u0〉,〈inc,read,0u0〉,〈inc,read,1u0〉) from the general gadget

in Figure 4g. – Create Counter Read(〈inc,read,u1〉,〈inc,read,0u1〉,〈inc,read,1u1〉) from the general gadget

in Figure 4h. – Create Counter Read(〈copy,read,u0〉,〈copy,read,0u0〉,〈copy,read,1u0〉) from the general gad-

get in Figure 4g. – Create Counter Read(〈copy,read,u1〉,〈copy,read,0u1〉,〈copy,read,1u1〉) from the general gad-

get in Figure 4h.

• For each i = 0,...,m − 2:

– Create Counter Read Msb(〈inc,read, bin(i, l)〉,〈copy,write,i + 1,1〉) from the general gadget

in Figure 4e if bit(i, m,1,l) = 0 or Figure 4f if bit(i, m,1,l) = 1.

• Create Counter Read Msb(〈inc,read, bin(m − 1,l)〉,〈inc,write,1〉) from the general gadget in Fig- ure 4e if bit(i, m,1,l) = 0 or Figure 4f if bit(i, m,1,l) = 1.

• For each i = 0,...,m − 1:

– Create Counter Read Msb(〈copy,read, bin(i, l)〉,〈copy,write, i,1〉) from the general gadget in

Figure 4e if bit(i, m,1,l) = 0 or Figure 4f if bit(i, m,1,l) = 1.

• For each i = 0,...,m − 1 and each j = 1,...,l − 1:

– Create Counter Write(〈copy,write, i, j〉,〈copy,write, i, j + 1〉) from the general gadget in Fig-

ure 4k if bit(i, m,1,j) = 0 or Figure 4l if bit(i, m,1,j) = 1.

• For each i = 1,...,l − 1:

3

– Create Counter Write(〈inc,write,i〉,〈inc,write,i + 1〉) from the general gadget in Figure 4k.

• For each i = 0,...,m − 1:

– Create Counter Write Msb(〈copy,write, i, l〉,〈copy,down z 0,1〉) from the general gadget in Fig-

ure 4i if bit(i, m,1,l) = 0 or Figure 4j if bit(i, m,1,l) = 1.

• Create Counter Write Msb(〈inc,write,l〉,〈inc,down z 0,1〉) from the general gadget in Figure 4i.

• For each i = 1,...,3l − 1:

– Create Down Column(〈inc,down z 0,i〉,〈inc,down z 0,i + 1〉) from the general gadget in Fig-

ure 1b. – Create Down Column(〈copy,down z 0,i〉,〈copy,down z 0,i + 1〉) from the general gadget in Fig-

ure 1b.

• Create Counter Return Column Start(〈inc,down z 0,3l〉,〈inc,down z 1,1〉) from the general gadget in Figure 4b.

• Create Counter Return Column Start(〈copy,down z 0,3l〉,〈copy,down z 1,1〉) from the general gad- get in Figure 4b.

• For each i = 1,...,l − 1:

– Create Counter Return Column(〈inc,down z 1,i〉,〈inc,down z 1,i + 1〉) from the general gad-

get in Figure 4c. – Create Counter Return Column(〈copy,down z 1,i〉,〈copy,down z 1,i + 1〉) from the general gad-

get in Figure 4c.

• Create Counter End(〈inc,down z 1,l〉,〈inc,start〉,〈elw,found〉) from the general gadget in Fig- ure 4d.

• Create Counter End(〈copy,down z 1,l〉 ,〈copy,start〉,〈lw,found〉) from the general gadget in Fig- ure 4d.

1.4 Left wall gadgets

The purpose of the Left Wall gadget is to block the construction of additional Counter units for the next row, forcing them to produce a Left Wall or End Left Wall unit. The preceding counter row will have been blocked by the preceding Left Wall unit or the Seed unit. Additionally, the Left Wall unit initiates a Return gadget with its east-facing glue.

1.5 Return gadget

The Return gadget ends with a Guess gadget on its east side which produces more Return gadgets in that direction, passing over the tiles that were left by the Counter gadgets until this row of Return gadgets reaches the preceding Right Wall gadget and is forced to initiate a Right Wall Foundation gadget. The Right Wall Foundation gadget simply initiates another Right Wall gadget.

This entire process repeats itself until an End Left Wall gadget is produced, at which point all Counter gadgets will be outputting 0.

4

(b) Counter Return Column Start

(c) Counter Return Column

(a) Counter Start

(d) Counter End

(a) Left Wall Foundation

(b) Left Wall End

Figure 5: The Left Wall gadget unit.

5

(l) Counter Write 1

Figure 4: The Counter gadget unit.

(e) Counter Read Msb 0

(f) Counter Read Msb 1

(h) Counter Read 1

(g) Counter Read 0

(i) Counter Write Msb 0

(j) Counter Write Msb 1

(k) Counter Write 0

(c) End Left Wall End

Figure 7: The End Left Wall gadget unit

1.6 End left wall gadgets

The End Left Wall gadget is made tall enough to block all future Shingle Tiles, and then initiates an End Return gadget from its east-facing glue.

1.7 End return gadgets

The Guess gadget on the east end of End Return gadget initiates either another End Return gadget or the Roof gadget. The End Return gadget is also extended northward in order to cover any empty tile spaces that would have been filled by the, never to return, counter row.

(e) End Return Row End

Figure 8: The End Return Row gadget unit.

1.8 Roof gadgets

The Roof gadget contains a hard-coded tile column that reaches above the protruding tiles from the last counter row, that then extends the assembly to a height of N. Each tile in the vertical column that extends past the counter row has a west-facing glue. The west-facing glue accepts a Shingle Tile which extend the roof westward until blocked by the End Left Wall gadget. The east-facing glue on the Roof gadget’s north-most column tile is appended until it reaches (k − 1,N − 1), which is the northeast corner of the construction’s rectangle shape. Each tile that extends the roof eastward has a south-facing glue that accepts a Drop Tile. The Drop Tiles will replicate southward until blocked by a Right Wall gadget or the “lip” on the Seed gadget.

6

(a) End Left Wall Foundation

(b) End Return Row Column

(a) End Return Row Start

Figure 6: The Return Row gadget.

(c) End Return Row Msb Filler

(d) End Return Row Filler

(b) End Left Wall Cap

(a) Roof Foundation

(e) Roof Cap 2

(b) Roof Column

Figure 9: The Roof gadget unit.

1.9 Overview

An entire 2-digit, ternary example of the counter is illustrated in Figure ??.

1.10 Tile complexity

The tile complexity for each gadget is O (...), but since we have 2m Counter gadgets, the consolidation of each gadget set brings the complexity of the entire construction to O (...). D

7

(d) Roof Cap 1

(c) Roof Cap 0